Exact Controllability for Stochastic Transport Equations*

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Abstract

This paper is addressed to studying the exact controllability for stochastic transport equations by two controls. One is a boundary control in the drift term and the other is an internal control in the diffusion term. By means of the standard duality argument, the control problem is converted into an observability problem for backward stochastic transport equations, and the desired observability estimate is obtained by a global Carleman estimate. At last, we give some results about the lack of exact controllability which show the action of two controls is necessary. To some extent, this shows that the control problems for stochastic PDEs differ from its deterministic counterpart.

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1 Introduction

Let $T > 0$ and $G \subset \mathbb{R}^d (d \in \mathbb{N})$ a strictly convex bounded domain with the C^1 boundary Γ. Denote by $\nu(x) = (\nu^1(x), \cdots, \nu^d(x))$ the unit outward normal vector to Γ at x. Let $\bar{x}_1, \bar{x}_2 \in \Gamma$ such that

$$|\bar{x}_1 - \bar{x}_2|_{\mathbb{R}^d} = \max_{x_1, x_2 \in \overline{G}} |x_1 - x_2|_{\mathbb{R}^d}.$$

Without loss of generality, we assume that $0 \in G$ and $0 = \bar{x}_1 + \bar{x}_2$. Put $R = \max_{x \in \Gamma} |x|_{\mathbb{R}^d}$. Let

$$S^{d-1} \overset{\triangle}{=} \{x : x \in \mathbb{R}^d, |x|_{\mathbb{R}^d} = 1\}.$$

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Denote by

$$\Gamma_{-S} = \{(x,U) \in \Gamma \times S^{d-1} : U \cdot \nu(x) \le 0\} \text{ and } \Gamma_{+S} = \big(\Gamma \times S^{d-1}\big) \setminus \Gamma_{-S}.$$

Let us define a Banach space $L^2_w(\Gamma_{-S})$ as the completion of all $h \in C_0^\infty(\Gamma_- \times S^{d-1})$ with the norm

$$|h|_{L^2_w(\Gamma_{-S})} \triangleq \Big(-\int_{\Gamma_{-S}} U \cdot \nu |h|^2 d\Gamma dS \Big)^{\frac{1}{2}},$$

where dS denotes the Lebesgue measure on S^{d-1}. Clearly, $L^2_w(\Gamma_{-S})$ is a Hilbert space with the inner product $(\cdot,\cdot)_{L^2_w(\Gamma_{-S})}$ given by

$$(h_1,h_2)_{L^2_w(\Gamma_{-S})} = -\int_{\Gamma_{-S}} U \cdot \nu h_1 h_2 d\Gamma dS,$$

and $L^2(\Gamma_{-S})$ is dense in $L^2_w(\Gamma_{-S})$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t\ge0}$ is defined such that $\{\mathcal{F}_t\}_{t\ge0}$ is the natural filtration generated by $\{B(t)\}_{t\ge0}$, augmented by all the P-null sets in $\mathcal{F}$. Let H be a Banach space. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\ge0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\ge0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\ge0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];H)}) < \infty$(similarly, one can define $L^2_{\mathcal{F}}(\Omega; C^k([0,T];H))$ for any positive integer k). All of the above spaces are endowed with the canonical norm.

The main purpose of this paper is to study the exact controllability of the following controlled linear forward stochastic transport equation:

$$\begin{cases} dy + U\cdot\nabla y dt = \Big[a_1 y + \displaystyle\int_{S^{d-1}} a_2(t,x,U,V) y(t,x,V) dS + f\Big]dt \\ \qquad\qquad\qquad +\big[a_3 y + v\big] dB(t) & \text{in } (0,T)\times G\times S^{d-1}, \\ y = u & \text{on } (0,T)\times \Gamma_{-S}, \\ y(0) = y_0 & \text{in } G\times S^{d-1}. \end{cases} \tag{1.1}$$

Here and in the sequel, ∇ denotes the gradient operator with respect to x,

$$\begin{cases} y_0 \in L^2(G\times S^{d-1}), \\ a_1 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1})), \\ a_2 \in L^\infty_{\mathcal{F}}(\Omega; C^1([0,T]; C^1(\overline{G}\times S^{d-1}\times S^{d-1}))), \\ a_3 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1})), \\ f \in L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1})). \end{cases}$$

The boundary control function $u \in L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))$, and the internal control function $v \in L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))$.

We begin with the definition for the solution to the system (1.1).

Definition 1.1 *A solution to the system* (1.1) *is a process* $y \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G\times S^{d-1})))$ *such that for every* $\tau \in [0,T]$ *and every*

$$\phi \in C^1(\overline{G} \times S^{d-1}), \quad \phi = 0 \text{ on } \Gamma_{+S},$$

it holds that

$$\begin{aligned}
&\int_G \int_{S^{d-1}} y(\tau, x, U)\phi(x, U) dS dx - \int_G \int_{S^{d-1}} y_0(x, U)\phi(x, U) dS dx \\
&-\int_0^\tau \int_G \int_{S^{d-1}} y(s, x, U) U \cdot \nabla\phi(x, U) dS dx ds + \int_0^\tau \int_{\Gamma_{-S}} u(s, x, U)\phi(x, U) U \cdot \nu d\Gamma_{-S} ds \\
= &\int_0^\tau \int_G \int_{S^{d-1}} \big[a_1(s, x, U) y(s, x, U) + f(s, x, U)\big]\phi(x, U) dS dx ds \\
&+ \int_0^\tau \int_G \int_{S^{d-1}} \Big[\int_{S^{d-1}} a_2(s, x, U, V) y(t, x, V) dS\Big]\phi(x, U) dS dx ds \\
&+ \int_0^\tau \int_G \int_{S^{d-1}} \big[a_3(s, x, U) y(s, x, U) + v(s, x, U)\big]\phi(x, U) dS dx dB(s), \quad P\text{-a.s.}
\end{aligned} \tag{1.2}$$

In Section 2, we will prove the following result.

Proposition 1.1 *For each* $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, *the system* (1.1) *admits a unique solution* y. *Further, there is a constant* $C > 0$ *such that for every* $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G\times S^{d-1}))$, *it holds that*

$$\begin{aligned}
&|y|_{L^2(\Omega; C([0,T]; L^2(G\times S^{d-1})))} \\
\le\, &e^{Cr_1}\big(\mathbb{E}|y_0|_{L^2(G\times S^{d-1})} + |f|_{L^2_{\mathcal{F}}(0,T; L^2(G\times S^{d-1}))} + |u|_{L^2_{\mathcal{F}}(0,T; L^2_w(\Gamma_{-S}))} + |v|_{L^2_{\mathcal{F}}(0,T; L^2(G\times S^{d-1}))}\big).
\end{aligned} \tag{1.3}$$

Here

$$r_1 = |a_1|^2_{L^\infty_{\mathcal{F}}(0,T; L^\infty(G\times S^{d-1}))} + |a_2|_{L^\infty_{\mathcal{F}}(\Omega; C^1([0,T]; C^1(\overline{G}\times S^{d-1}\times S^{d-1})))} + |a_3|_{L^\infty_{\mathcal{F}}(0,T; L^\infty(G\times S^{d-1}))} + 1.$$

Now we give the formulation for the exact controllability of the system (1.1).

Definition 1.2 *System* (1.1) *is said to be exactly controllable at time* T *if for every initial state* $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G \times S^{d-1}))$ *and every* $y_1 \in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1}))$, *one can find a pair of controls* $(u, v) \in L^2_{\mathcal{F}}(0, T; L^2_w(\Gamma_{-S})) \times L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$ *such that the solution* y *of the system* (1.1) *satisfies that* $y(T) = y_1$ *in* $L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1}))$.

We have the following result for the exact controllability of the system (1.1).

Theorem 1.1 *System* (1.1) *is exactly controllable at time* T, *provided that* $T > 2R$.

We put two controls on the system. Moreover, the control v acts on the whole domain. Compared with the deterministic transport solution, it seems that our choice of controls is too restrictive. One may consider the following four weaker cases for designing the control.

1. Only one control is acted on the system, that is, $u = 0$ or $v = 0$ in (1.1).
2. Neither u nor v is zero. But $v = 0$ in $(0, T) \times G_0$, where G_0 is a nonempty open subset of G.
3. Two controls are acted on the system. But both of them are in the drift term.
4. The time $T < 2R$.

It is easy to see the lack of exact controllability for the fourth case. Indeed, if the system (1.1) is exactly controllable at some time $T < 2R$, then one can get the exact controllability of a deterministic transport equation on G at time T with a boundary control acted on Γ_{-S}, which is obviously untrue. For the other three cases, according to the controllability result for deterministic transport equations, it seems that the system should be exact controllable. However, it is not the truth. Indeed, we have the following result.

We establish the exact controllability of the system (1.1) by employing two controls. As we all know, deterministic transport equations are exactly controllable by utilizing only one boundary control. It is natural to ask whether one control is enough to achieve the exact controllability for stochastic transport equations. Indeed, we have the following result.

Theorem 1.2 *If $u \equiv 0$ or $v \equiv 0$ in the system* (1.1), *then the system* (1.1) *is not exactly controllable at any time T.*

Theorem 1.2 shows that two control is necessary for exact controllability. However, the control v in the diffusion term ia acted on the whole domain G, one may expect to get the exact controllability of (1.1) with v supported in a subdomain of G. However, this wish cannot be true.

Theorem 1.3 *Let G_0 be a nonempty open subset of G. If $v \equiv 0$ $(0, T) \times G_0$, then the system* (1.1) *is not exactly controllable at any time T.*

For the third case, we consider the following controlled equation:

$$
\begin{cases}
dy + U \cdot \nabla y dt = \Big[a_1 y + \displaystyle\int_{S^{d-1}} a_2(t, x, U, V) y(t, x, V) dS + f + \ell\Big] dt \\
\qquad\qquad\qquad +a_3 y dB(t) & \text{in } (0, T) \times G \times S^{d-1}, \\
y = u & \text{on } (0, T) \times \Gamma_{-S}, \\
y(0) = y_0 & \text{in } G \times S^{d-1}.
\end{cases}
\tag{1.4}
$$

Here $\ell \in L^2_{\mathcal{F}}(0, T; L^2(G))$ is a control.

Theorem 1.4 *The system* (1.4) *is not exact controllable for any $T > 0$.*

In order to prove Theorem 1.1, we make use of the classical duality argument. We obtain the exact controllability of the system (1.1) by establishing an observability estimate for the following backward stochastic transport equation:

$$
\begin{cases}
dz + U\cdot\nabla z dt = \Big[b_1 z + \displaystyle\int_{S^{d-1}} b_2(t,x,V,U)z(t,x,V) + b_3 Z\Big]dt \\
\qquad\qquad\qquad + (b_4 z + Z)dB(t) & \text{in } (0,T)\times G\times S^{d-1},\\
z = 0 & \text{on } (0,T)\times\Gamma_+,\\
z(T) = z_T & \text{in } G\times S^{d-1}.
\end{cases}
\tag{1.5}
$$

Here

$$
\begin{cases}
z_T \in L^2(\Omega,\mathcal{F}_T,P;L^2(G\times S^{d-1})),\\
b_1 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1})),\\
b_2 \in L^\infty_{\mathcal{F}}(\Omega;C^1([0,T];C^1(\overline{G}\times S^{d-1}\times S^{d-1}))),\\
b_3 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1})),\\
b_4 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1})).
\end{cases}
$$

Before studying the observability estimate for the equation (1.5), we first give the definition of the solution to it.

Definition 1.3 *A solution to the equation* (1.5) *is a pair of stochastic processes*

$$
(z,Z)\in L^\infty_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))\times L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))
$$

such that for every $\psi\in C_0^\infty(G\times S^{d-1})$ and a.e. $(\tau,\omega)\in[0,T]\times\Omega$, it holds that

$$
\begin{aligned}
&\int_G\int_{S^{d-1}} z_T(x,U)\psi(x,U)dSdx - \int_G\int_{S^{d-1}} z(\tau,x,U)\psi(x,U)dSdx\\
&-\int_\tau^T\int_G\int_{S^{d-1}} z(s,x,U)U\cdot\nabla\psi(x,U)dSdxds\\
=&\int_\tau^T\int_G\int_{S^{d-1}}\big[b_1(s,x,U)z(s,x,U)+b_3(s,x,U)Z(s,x,U)\big]\psi(x)dSdxds\\
&+\int_\tau^T\int_G\int_{S^{d-1}}\Big[\int_{S^{d-1}} b_2(t,x,V,U)z(t,x,V)dS\Big]\psi(x,U)dSdxds\\
&\quad+\int_\tau^T\int_G\int_{S^{d-1}}\big[b_4(s,x,U)z(s,x,U)+Z(s,x,U)\big]\psi(x,U)dSdxdB(s).
\end{aligned}
\tag{1.6}
$$

In Section 2, we will establish the following result.

Proposition 1.2 *For any $z_T\in L^2(\Omega,\mathcal{F}_T,P;L^2(G\times S^{d-1}))$, the equation* (1.5) *admits a unique solution (z,Z). Moreover, (z,Z) satisfies that*

$$
|z|_{L^\infty_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}+|Z|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}\le e^{Cr_2}|z_T|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G\times S^{d-1}))},
\tag{1.7}
$$

where

$$r_2 \triangleq \sum_{i=1,i\neq 2}^{4} |b_i|^4_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1}))} + |b_2|_{L^\infty_{\mathcal{F}}(\Omega;C^1([0,T];C^1(\overline{G}\times S^{d-1}\times S^{d-1})))} + 1.$$

Now we give the definition of the exact observability for the equation (1.5).

Definition 1.4 *Equation* (1.5) *is exactly observable at time* T *if there exists a constant* $\mathcal{C}(b_1,b_2,b_3,b_4)$ *such that all solutions of the equation* (1.5) *satisfy that*

$$|z_T|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G\times S^{d-1}))} \leq \mathcal{C}(b_1,b_2,b_3,b_4)\big(|z|_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))} + |Z|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}\big). \quad (1.8)$$

The solution z only belongs to $L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))$, hence, it is not obvious that $z|_{\Gamma_-}$ belongs to $L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_-))$. Fortunately, it is true by the following result.

Proposition 1.3 *Let* $(z,Z)\in L^\infty_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))\times L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))$ *solves the equation* (1.5) *with the terminal state* z_T*. Then*

$$|z|^2_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))} \leq e^{Cr_2}\mathbb{E}|z_T|^2_{L^2(G\times S^{d-1})}.$$

Remark 1.1 *The fact that* $z|_{\Gamma_{-S}}\in L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))$ *is sometimes called a hidden regularity property. It does not follow directly from the classical trace theorem of Sobolev space.*

It follows from Proposition 1.3 that $|z|^2_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))}$ makes sense. Now we give the result for the exact observability of the equation (1.5).

Theorem 1.5 *If* $T>2R$*, then the equation* (1.5) *is exactly observable at time* T*.*

In spite of its simple linear form, the transport equation governs almost every diffusion processes (see [7] for example). Moreover, it is a linearized Boltzmann equation, and it is related to the equations of fluid dynamics such as the Euler and the Navier-Stokes equations. It is desired to study the stochastic transport equations since it is an model when the system governed by the transport equation is perturbed by stochastic influence. The stochastic transport equation is extensively studied now (see [3, 5] for example).

The controllability problems for linear and nonlinear deterministic transport equations are well studied in the literature (see [1, 6, 8, 9, 12] and the references cited therein). On the contrast, to our best knowledge, there is no published paper addressed to the controllability of the stochastic transport equation.

Generally speaking, there are three methods to get the exact controllability of the deterministic transport equation. The first and most straightforward one is utilizing the explicit formula of the solution. By this method, for some simple transport equations, one can explicitly give a control steering the system from every given initial state to each given final state, provided that the time is large enough. On one hand, it seems that this method cannot be used to solve our problem since we do not know the explicit formula for the solution to the system (1.1). On the other hand, we borrow this idea to prove our negative result (Theorem

1.2). The second one is the extension method. This method was first introduced in [13] for obtaining the exact controllability of wave equations. It is useful to solve the exact controllability problem for many hyperbolic equations. However, it seems that it is only valid for time reversible systems. The third and most popular method is based on the duality between controllability and observability. Solving the exact controllability problem is transformed to establishing some suitable observability estimate, and the desired observability estimate is obtained by a global Carleman estimate (see [9] for example).

Similar to the deterministic setting, we shall use a stochastic version of the global Carleman estimate to derive the inequality (1.4). For this, we borrow some idea from the proof of the observability estimate for deterministic transport equations (see [9] for example). However, the stochastic setting will produce some more undesired terms in the computation. We cannot simply mimic the method in [9] to solve our problem.

In the literature, in order to obtain the observability, people usually combine the "Carleman estimate" and "Energy estimate"(see [9, 14] for example). In this paper, we deduce the inequality (1.4) by our global Carleman estimate directly. Indeed, our method even provide a simpler proof for the observability estimate for deterministic transport equation.

The rest of this paper is organized as follows. In Section 2, we first present the proofs of Proposition 1.1-1.3. Then we also give a weighted identity, which will play an important role in establishing the global Carleman estimate for the equation (1.5). Further, we give a lemma which is utilized to show the lack of exact controllability results. Section 3 is addressed to proving Theorem 1.5 and Section 4 is devoted to proving Theorem 1.1. At last, in Section 5, we prove Theorem 1.2–1.4.

2 Some preliminaries

In this section, we present some preliminary results. We first establish the well-posedness of the equation (1.1) and (1.5). Then, we prove a fundamental weighted identity which plays a key role in proving the observability estimate of the equation (1.5). At last, we give a lemma that is used to show the lack of exact controllability.

Proof of Proposition 1.1: **The existence of the solution.** Let us first deal with the case in which

$$\left\{\begin{array}{l} u\in L^2_{\mathcal{F}}(\Omega;C^2([0,T];H^1_{\Gamma_{+S}}(G\times S^{d-1})\cap H^2(G\times S^{d-1}))),\ u(0,\cdot,\cdot)=0 \text{ on } \Gamma_{-S},\ P\text{-a.s.},\\ y_0\in L^2(\Omega,\mathcal{F}_0,P;H^1(G\times S^{d-1})) \text{ and } y_0=0 \text{ on } \Gamma_{-S},\quad P\text{-a.s.},\\ f,v\in L^2_{\mathcal{F}}(0,T;H^1_0(G\times S^{d-1})). \end{array}\right. \tag{2.1}$$

Here $H^1_{0,\Gamma_{+S}}(G\times S^{d-1}))\stackrel{\triangle}{=}\{u:\ u\in H^1(G\times S^{d-1}),\ u=0 \text{ on } \Gamma_{+S}\}$.

Let us consider the following equation:

$$\begin{cases} dw + U\cdot\nabla w dt = \Big(a_1 w + \displaystyle\int_{S^{d-1}} a_3(t,x,U,V)w(t,x,V)dS + \tilde f\Big)dt & \\ \qquad\qquad\qquad +(a_2 w + v)dB(t) + a_2 u dB(t) & \text{in } (0,T)\times G\times S^{d-1}, \\ w(t,0)=0 & \text{on } (0,T)\times\Gamma_{-S}, \\ w(0)=y_0 & \text{in } G\times S^{d-1}. \end{cases} \tag{2.2}$$

Here

$$\tilde f = -u_t - U\cdot\nabla u + a_1 u + \int_{S^{d-1}} a_3(t,x,U,V)u(t,x,V)dS + f.$$

Clearly, $\tilde f \in L^2_{\mathcal F}(0,T;H^1(G\times S^{d-1}))$. Define an unbounded operator A on $L^2(G\times S^{d-1})$ as follows:

$$\begin{cases} D(A) = \left\{h\in H^1(G\times S^{d-1}):\ h=0 \text{ on } \Gamma_{-S}\right\}, \\ Ah = -U\cdot\nabla h, \quad \forall\, h\in D(A). \end{cases}$$

It is an easy matter to see that $D(A)$ is dense in $L^2(G\times S^{d-1})$ and A is closed. Furthermore, A satisfies the property that for every $h\in D(A)$,

$$(Ah,h)_{L^2(G\times S^{d-1})} = -\int_G\int_{S^{d-1}} hU\cdot\nabla h dSdx = -\int_{\Gamma_{+S}} U\cdot\nu|h|^2 d\Gamma \le 0.$$

One can easily check that the adjoint operator of A is

$$\begin{cases} D(A^*) = \left\{h\in H^1(G\times S^{d-1}):\ h=0 \text{ on } \Gamma_{+S}\right\}, \\ A^*h = U\cdot\nabla h, \quad \forall\, h\in D(A^*). \end{cases}$$

For every $h\in D(A^*)$, it holds that

$$(A^*h,h)_{L^2\times S^{d-1}} = \int_G\int_{S^{d-1}} hU\nabla h dx = \int_{\Gamma_{-S}} a_0\cdot\nu|h|^2 d\Gamma \le 0.$$

Then, we find that both A and A^* are dissipative operators. Recalling that $D(A)$ is dense in $L^2(G\times S^{d-1})$ and A is closed. We know that A generates a C_0 semigroup $\{S(t)\}_{t\ge 0}$ on $L^2(G\times S^{d-1})$ (see [4, Page 84] for example). Therefore, by classical theory for stochastic partial differential equations(see [2, Chapter 6]), we know that the system (2.2) admits a unique solution

$$w\in L^2_{\mathcal F}(\Omega; C([0,T];L^2(G\times S^{d-1})))\cap L^2_{\mathcal F}(0,T;D(A))$$

such that

$$
\begin{aligned}
&\int_G \int_{S^{d-1}} w(\tau,x)\phi(x)dSdx - \int_G \int_{S^{d-1}} y_0(x)\phi(x)dSdx \\
&-\int_0^\tau \int_G \int_{S^{d-1}} w(s,x)U\cdot\nabla\phi(x)dSdxds \\
=&\int_0^\tau \int_G \int_{S^{d-1}} \big[a_1(s,x,U)w(s,x,U)+\tilde f(s,x,U)\big]\phi(x,U)dSdxds \\
&+\int_0^\tau \int_{S^{d-1}} \Big[\int_G \int_{S^{d-1}} a_2(t,x,U,V)w(t,x,V)dS\Big]\phi(x,U)dSdxds \\
&+\int_0^\tau \int_G \int_{S^{d-1}} \big\{a_3(s,x,U)\big[w(s,x,U)+u(s,x,U)\big]+v(s,x,U)\big\}\phi(x,U)dSdxdB(s), \\
&\qquad P\text{-a.s., for any } \phi\in C^1(\overline{G}\times S^{d-1}),\quad \phi=0 \text{ on } \Gamma_{+S}, \text{ and } \tau\in[0,T].
\end{aligned}
\tag{2.3}
$$

Let

$$
y(t,x,U)=w(t,x,U)+u(t,x,U),\quad \forall (t,x,U)\in[0,T]\times G\times S^{d-1}.
$$

Clearly,

$$
y\in L^2_{\mathcal F}(\Omega;C([0,T];L^2(G\times S^{d+1})))\cap L^2_{\mathcal F}(0,T;D(A)).
$$

From the equality (2.3), we know that y satisfies

$$
\begin{aligned}
&\int_G \int_{S^{d-1}} y(\tau,x,U)\phi(x,U)dSdx - \int_G \int_{S^{d-1}} y_0(x,U)\phi(x,U)dSdx \\
&-\int_0^\tau \int_G \int_{S^{d-1}} y(s,x,U)U\cdot\nabla\phi(x,U)dSdxds + \int_0^\tau \int_G \int_{S^{d-1}} u(s,x,U)U\cdot\nabla\phi(x,U)dSdxds \\
=&\int_0^\tau \int_G \int_{S^{d-1}} \big[a_1(s,x,U)y(s,x,U)+f(s,x,U)-u_t(s,x,U)-U\cdot\nabla u(s,x,U)\big]\phi(x)dSdxds \\
&+\int_0^\tau \int_G \int_{S^{d-1}} \Big[\int_{S^{d-1}} a_2(t,x,U,V)y(t,x,V)dS\Big]\phi(x,U)dSdxds \\
&+\int_0^\tau \int_G \int_{S^{d-1}} \big[a_3(s,x,U)y(s,x,U)+v(s,x,U)\big]\phi(x,U)dSdxdB(s), \\
&\qquad P\text{-a.s., for any } \phi\in C^1(\overline{G}\times S^{d-1}),\quad \phi=0 \text{ on } \Gamma_{+S}, \text{ and } \tau\in[0,T].
\end{aligned}
$$

Utilizing integration by parts again, we see

$$
\begin{aligned}
&\int_G \int_{S^{d-1}} y(\tau,x,U)\phi(x,U)dSdx - \int_G \int_{S^{d-1}} y_0(x,U)\phi(x,U)dSdx \\
&- \int_0^\tau \int_G \int_{S^{d-1}} y(s,x,U) U\cdot\nabla\phi(x,U)dSdxds \\
= &\int_0^\tau \int_G \int_{S^{d-1}} \big[a_1(s,x,U)y(s,x,U) + f(s,x,U) - u_t(s,x,U)\big]\phi(x)dSdxds \\
&+ \int_0^\tau \int_G \int_{S^{d-1}} \Big[\int_{S^{d-1}} a_2(t,x,U,V)y(t,x,V)dS\Big]\phi(x,U)dSdxds \\
&+ \int_0^\tau \int_G \int_{S^{d-1}} \big[a_3(s,x,U)y(s,x,U) + v(s,x,U)\big]\phi(x,U)dSdxdB(s) \\
&\qquad P\text{-a.s., for any } \phi\in C^1(\overline{G}\times S^{d-1}), \quad \phi=0 \text{ on } \Gamma_{+S}, \text{ and } \tau\in[0,T].
\end{aligned}
\tag{2.4}
$$

Therefore, y is a solution to the system (1.1) under the assumption (2.1). Furthermore, by means of Itô's formula, we know that

$$
\begin{aligned}
&\mathbb{E}|y(t)|^2_{L^2(G\times S^{d-1})} \\
= &\mathbb{E}|y_0|^2_{L^2(G\times S^{d-1})} - 2\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} yU\nabla y dSdxds + 2\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} \Big[\int_{S^{d-1}} a_2 y dS\Big] y dSdxds \\
&+\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} \big[2a_1y^2 + 2fy + (a_3y+v)^2\big]dSdxds \\
\le &\mathbb{E}|y_0|^2_{L^2(G\times S^{d-1})} - \mathbb{E}\int_0^t \int_{\Gamma_{-S}} U\cdot\nu u^2 d\Gamma ds + 2\mathbb{E}\int_0^t \int_G |a_2|_{C(\overline{G}\times S^{d-1}\times S^{d-1})}\int_{S^{d-1}} y^2 dSdxds \\
&+2\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} \big[a_1y^2 + y^2 + f^2 + a_3^2y^2 + v^2\big]dSdxds \\
\le &\mathbb{E}|y_0|^2_{L^2(G\times S^{d-1})} + 2\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} \big(a_1 + a_3^2 + 1\big)y^2 dSdxds - \mathbb{E}\int_0^t \int_{\Gamma_{-S}} U\cdot\nu u^2 d\Gamma_{-S} ds \\
&+2\mathbb{E}\int_0^t \int_G |a_2|_{C(\overline{G}\times S^{d-1}\times S^{d-1})}\int_{S^{d-1}} y^2 dSdxds + 2\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} \big(f^2+v^2\big)dSdxds.
\end{aligned}
\tag{2.5}
$$

This, together with Gronwall's inequality, implies that

$$
\begin{aligned}
&|y|_{L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G\times S^{d-1})))} \\
\le &e^{Cr_1}\big(\mathbb{E}|y_0|_{L^2(G\times S^{d-1})} + |u|_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))} + |f|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))} + |v|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}\big).
\end{aligned}
\tag{2.6}
$$

By a similar argument, we can show that if

$$
\begin{aligned}
(\tilde{y}_0,\tilde{u},\tilde{f},\tilde{v}) \in L^2(\Omega,\mathcal{F}_0,P;D(A)) &\times L^2_{\mathcal{F}}(\Omega,C^2([0,T];H^1_{\Gamma_{+S}}(G\times S^{d-1}))) \\
&\times L^2_{\mathcal{F}}(0,T;H^1_0(G\times S^{d-1})) \times L^2_{\mathcal{F}}(0,T;H^1_0(G\times S^{d-1}))
\end{aligned}
$$

and

$$(\bar{y}_0, \bar{u}, \bar{f}, \bar{v}) \in L^2(\Omega, \mathcal{F}_0, P; D(A)) \times L^2_{\mathcal{F}}(\Omega, C^2([0,T]; H^1_{\Gamma_{+S}}(G \times S^{d-1})))$$
$$\times L^2_{\mathcal{F}}(0,T; H^1_0(G \times S^{d-1})) \times L^2_{\mathcal{F}}(0,T; H^1_0(G \times S^{d-1})),$$

then we can find corresponding solutions

$$\tilde{y}, \bar{y} \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G \times S^{d-1}))) \cap L^2_{\mathcal{F}}(0,T; D(A))$$

such that

$$\begin{aligned}
&|\tilde{y} - \bar{y}|_{L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G\times S^{d-1})))} \\
\leq\ & e^{Cr_1}\big(\mathbb{E}|\tilde{y}_0 - \bar{y}_0|_{L^2(G\times S^{d-1})} + |\tilde{u} - \bar{u}|_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))} + |\tilde{f} - \bar{f}|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))} \\
&+ |\tilde{v} - \bar{v}|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}\big).
\end{aligned}$$

Now let $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, $u \in L^2_{\mathcal{F}}(0,T; L^2_w(\Gamma_{-S}))$, $f, v \in L^2_{\mathcal{F}}(0,T; L^2(G \times S^{d-1}))$. Let us choose

$$\begin{aligned}
&\{y^n_0\}_{n=1}^{+\infty} \subset L^2(\Omega, \mathcal{F}_0, P; D(A)), \quad \{u^n\}_{n=1}^{+\infty} \subset L^2_{\mathcal{F}}(\Omega, C^2([0,T]; H^1_{\Gamma_{+S}}(G \times S^{d-1}))), \\
&\{f^n\}_{n=1}^{+\infty} \subset L^2_{\mathcal{F}}(0,T; H^1_0(G \times S^{d-1})), \quad \{v^n\}_{n=1}^{+\infty} \subset L^2_{\mathcal{F}}(0,T;\ H^1_0(G \times S^{d-1})),
\end{aligned}$$

such that

$$\begin{cases}
\lim\limits_{n\to\infty} y^n_0 = y_0 \text{ in } L^2(\Omega, \mathcal{F}_0, P; L^2(G \times S^{d-1})); \\
\lim\limits_{n\to\infty} u^n = u \text{ in } L^2_{\mathcal{F}}(0,T; L^2_w(\Gamma_{-S})); \\
\lim\limits_{n\to\infty} f^n = f \text{ in } L^2_{\mathcal{F}}(0,T; L^2(G \times S^{d-1})); \\
\lim\limits_{n\to\infty} v^n = v \text{ in } L^2_{\mathcal{F}}(0,T; L^2(G \times S^{d-1})).
\end{cases} \tag{2.7}$$

For every given (y^n_0, u^n, f^n, v^n), by the argument above, we know that there is a unique solution $y_n(\cdot,\cdot)$ to the system (1.1), which satisfies

$$\begin{aligned}
&\int_G \int_{S^{d-1}} y_n(\tau, x, U)\phi(x) dS dx - \int_G \int_{S^{d-1}} y^n_0(x, U)\phi(x, U) dS dx \\
&+ \int_0^\tau \int_G \int_{S^{d-1}} y_n(s, x, U) U \cdot \nabla\phi(x, U) dS dx ds \\
&- \int_0^\tau \int_{\Gamma_{-S}} U \cdot \nu u^n(s, x, U)\phi(x, U) d\Gamma_{-S} ds \\
=& \int_0^\tau \int_G \int_{S^{d-1}} \big[a_1(s, x, U) y_n(s, x, U) + f(s, x, U) - u_t(s, x, U)\big]\phi(x) dS dx ds \\
&+ \int_0^\tau \int_G \int_{S^{d-1}} \Big[\int_{S^{d-1}} a_2(t, x, U, V) y_n(t, x, V) dS\Big]\phi(x, U) dS dx ds \\
&+ \int_0^\tau \int_G \int_{S^{d-1}} \big[a_3(s, x, U) y_n(s, x, U) + v(s, x, U)\big]\phi(x, U) dS dx dB(s), \\
&\qquad P\text{-a.s., for any } \phi \in C^1(\overline{G} \times S^{d-1}), \quad \phi = 0 \text{ on } \Gamma_{+S}, \text{ and } \tau \in [0,T],
\end{aligned} \tag{2.8}$$

and

$$
\begin{aligned}
&|y_n|_{L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G\times S^{d-1})))}\\
&\le e^{Cr_1}\big(\mathbb{E}|y_0^n|_{L^2(G\times S^{d-1})}+|u_n|_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))}+|f_n|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}+|v_n|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}\big).
\end{aligned}
\tag{2.9}
$$

Further, for any $m,n\in\mathbb{N}$, we have

$$
\begin{aligned}
&|y_n-y_m|_{L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G\times S^{d-1})))}\\
&\le e^{Cr_1}\big(\mathbb{E}|y_0^n-y_0^m|_{L^2(G\times S^{d-1})}+|u_n-u_m|_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))}+|f_n-f_m|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}\\
&\quad+|v_n-v_m|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}\big).
\end{aligned}
\tag{2.10}
$$

By means of (2.7) and the inequality (2.10), we find that $\{y_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in $L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G\times S^{d-1})))$. Hence, there exists a unique $y\in L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G\times S^{d-1})))$ such that

$$
y_n\to y \text{ in } L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G\times S^{d-1}))) \text{ as } n\to+\infty. \tag{2.11}
$$

Combining (2.8) and (2.11), we obtain that

$$
\begin{aligned}
&\int_G\int_{S^{d-1}}y(\tau,x,U)\phi(x,U)dSdx-\int_G\int_{S^{d-1}}y_0(x,U)\phi(x,U)dSdx\\
&-\int_0^\tau\int_G\int_{S^{d-1}}y(s,x,U)U\cdot\nabla\phi(x,U)dSdxds
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
&=\int_0^\tau\int_G\int_{S^{d-1}}\big[a_1(s,x,U)y(s,x,U)+f(s,x,U)-u_t(s,x,U)\big]\phi(x)dSdxds\\
&\quad+\int_0^\tau\int_G\int_{S^{d-1}}\Big[\int_{S^{d-1}}a_2(t,x,U,V)y(t,x,V)dS\Big]\phi(x,U)dSdxds\\
&\quad+\int_0^\tau\int_G\int_{S^{d-1}}\big[a_3(s,x,U)y(s,x,U)+v(s,x,U)\big]\phi(x,U)dSdxdB(s),\\
&\qquad P\text{-a.s., for any } \phi\in C^1(\overline{G}\times S^{d-1}),\quad \phi=0 \text{ on } \Gamma_{+S}, \text{ and } \tau\in[0,T].
\end{aligned}
$$

Further, from (2.9) and (2.11), we find that

$$
\begin{aligned}
&|y|_{L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G\times S^{d-1})))}\\
&\le e^{Cr_1}\big(\mathbb{E}|y_0|_{L^2(G\times S^{d-1})}+|u|_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))}+|f|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}+|v|_{L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))}\big).
\end{aligned}
\tag{2.13}
$$

Hence, y is a solution to the system (1.1).

The uniqueness of the solution. Consider the following equation:

$$
\begin{cases}
dy+U\cdot\nabla ydt=\Big[a_1y+\displaystyle\int_{S^{d-1}}a_2(t,x,U,V)y(t,x,V)dS\Big]dt\\
\qquad\qquad\qquad +a_3ydB(t) & \text{in } (0,T)\times G\times S^{d-1},\\
y=0 & \text{on } (0,T)\times\Gamma_{-S},\\
y(0)=0 & \text{in } G\times S^{d-1}.
\end{cases}
\tag{2.14}
$$

Let $y \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G \times S^{d-1})))$ be a solution to the equation (2.14). Then, by means of Itô's formula, we know that

$$
\begin{aligned}
&\mathbb{E}|y(t)|^2_{L^2(G\times S^{d-1})}\\
&= \mathbb{E}|y_0|^2_{L^2(G\times S^{d-1})} - 2\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} yU\nabla y dSdxds + 2\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} \Big[\int_{S^{d-1}} a_2 y dS\Big] y dSdxds\\
&\quad +\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} \big[2a_1 y^2 + (a_3 y + v)^2\big] dSdxds\\
&\leq \mathbb{E}|y_0|^2_{L^2(G\times S^{d-1})} + 2\mathbb{E}\int_0^t \int_G \int_{S^{d-1}} \big(a_1 + a_2^2 + 1\big) y^2 dSdxds\\
&\quad +2\mathbb{E}\int_0^t \int_G |a_2|_{C^1(G\times S^{d-1}\times S^{d-1})} \int_{S^{d-1}} y^2 dSdxds.
\end{aligned}
$$

This, together with Gronwall's inequality, implies that

$$
|y|_{L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G\times S^{d-1})))} = 0.
$$

Hence, we know that the equation (2.14) admits only one solution $y \equiv 0$, which concludes the uniqueness of the solution to the system (1.1). □

Next, we give a proof of Proposition 1.2.

Proof of Proposition 1.2: **The uniqueness of the solution.** Let us suppose that

$$
(z, Z) \in L^\infty_{\mathcal{F}}(0,T; L^2(G \times S^{d-1})) \times L^2_{\mathcal{F}}(0,T; L^2(G \times S^{d-1}))
$$

is a solution to (1.5) with $z(T) = 0$. Then, by Itô's formula, for a.e. $t \in [0,T]$, we find that

$$
\begin{aligned}
\mathbb{E}|z(t)|^2_{L^2(G\times S^{d-1})} &= 2\mathbb{E}\int_t^T \int_G \int_{S^{d-1}} \big[zU\cdot\nabla z - b_1 z^2 - b_2 zZ\big] dSdxdt\\
&\quad -\mathbb{E}\int_t^T \int_G \int_{S^{d-1}} (b_3 z + Z)^2 dSdxdt\\
&\leq Cr_2\mathbb{E}\int_t^T |z(s)|^2_{L^2(G\times S^{d-1})} ds.
\end{aligned}
\tag{2.15}
$$

Therefore, by virtue of Gronwall's inequality, we conclude that

$$
z = 0 \quad \text{in } L^2_{\mathcal{F}}(0,T; L^2(G)).
$$

From the definition of the solution to the equation (1.5), for any $\psi \in C_0^\infty(G \times S^{d-1})$, it holds that

$$
\int_\tau^T \big(b_2(s)Z(s), \psi\big)_{L^2(G\times S^{d-1})} ds + \int_\tau^T \big(Z(s), \psi\big)_{L^2(G\times S^{d-1})} dB(s) = 0.
$$

This, together with the uniqueness of the decomposition of the semi-martingale, leads to the fact that $(Z(\cdot), \psi)_{L^2(G\times S^{d-1})} \equiv 0$ in $[0,T]$, P-a.s. Hence

$$
Z = 0 \quad \text{in } L^2_{\mathcal{F}}(0,T; L^2(G \times S^{d-1})).
$$

The existence of the solution. We borrow some ideas from [15]. Let $\{e_i\}_{i=1}^{+\infty}$ be the eigenfunctions of the homogeneous Dirichlet Laplacian on $L^2(G \times S^{d-1})$ such that $|e_i|_{L^2(G\times S^{d-1})} = 1$ $(i = 1, 2, \cdots)$. Let

$$\begin{cases} \mathcal{A}^n = \big((U \cdot \nabla e_i, e_j)_{L^2(G\times S^{d-1})}\big)_{1\le i,j\le n}, \\ \mathcal{B}_k^n(t) = \big((b_k e_i, e_j)_{L^2(G\times S^{d-1})}\big)_{1\le i,j\le n} \text{ for } k = 1, 3, 4, \\ \mathcal{B}_2^n(t) = \big(((b_2, e_i)_{L^2(S^{d-1})}, e_j)_{L^2(G\times S^{d-1})}\big)_{1\le i,j\le n}. \end{cases}$$

It is an easy matter to see that $\mathcal{A}^n$ is a skew-adjoint matrix. Since $b_k \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G \times S^{d-1}))$ $(k = 1, 3, 4)$, we know that

$$\mathcal{B}_k^n \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{n\times n}) \text{ and } |\mathcal{B}_k^n|_{L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{n\times n})} \le |b_k|_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1}))}, \quad k = 1, 2, 3.$$

From $b_2 \in L^\infty_{\mathcal{F}}(\Omega; C^1([0, T]; C^1(\overline{G} \times S^{d-1} \times S^{d-1})))$, we find that

$$\mathcal{B}_2^n \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{n\times n}) \text{ and } |\mathcal{B}_2^n|_{L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{n\times n})} \le |b_2|_{L^\infty_{\mathcal{F}}(\Omega;C^1([0,T];C^1(\overline{G}\times S^{d-1}\times S^{d-1})))}.$$

By the classical theory of backward stochastic differential equations (see [10, Chapter 1] for example), we know that there is a unique

$$z_n = (z_{n1}, \cdots, z_{nn})^{\mathrm{T}} \in L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n))$$

and a unique

$$Z_n = (Z_{n1}, \cdots, Z_{nn})^{\mathrm{T}} \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n),$$

which solve the following equation:

$$\begin{cases} dz_n + \mathcal{A}^n z_n = (\mathcal{B}_1^n z_n + \mathcal{B}_2^n z_n + \mathcal{B}_3^n Z_n)\, dt + (\mathcal{B}_4^n z_n + Z_n)\, dB(t) \quad \text{in } [0, T], \\ z_n(T) = z_T^n. \end{cases} \tag{2.16}$$

Here $z_T^n = \big((z_T, e_1)_{L^2(G\times S^{d-1})}, \cdots, (z_T, e_n)_{L^2(G\times S^{d-1})}\big)^{\mathrm{T}}$. Moreover, (z_n, Z_n) satisfies

$$\begin{aligned} |(z_n, Z_n)|_{L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{R}^n))\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)} &\le e^{C\big(\sum_{k=1}^4 |\mathcal{B}_k^n|^2_{L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{n\times n})}+1\big)T} |z_T|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G\times S^{d-1}))} \\ &\le e^{Cr_2} |z_T|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G\times S^{d-1}))}. \end{aligned} \tag{2.17}$$

Let

$$\begin{cases} z^n = \displaystyle\sum_{i=1}^n z_{ni} e_i \in L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1})), \\ Z^n = \displaystyle\sum_{i=1}^n Z_{ni} e_i \in L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1})). \end{cases}$$

By virtue of the inequality (2.17), we find that the sequence $\{z^n\}_{n=1}^{+\infty}$ is uniformly bounded in $L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$. Hence, we know that there is a subsequence $\{z^{n_j}\}_{j=1}^{+\infty}$ of $\{z^n\}_{n=1}^{+\infty}$, which is weakly convergent in $L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$. Utilizing the inequality (2.17) again,

we see $\{Z^{n_j}\}_{j=1}^{+\infty}$ is also uniformly bounded in $L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))$. Hence, we know that there is a subsequence $\{Z^{n_k}\}_{k=1}^{+\infty}$ of it, which weakly converges in $L^2_{\mathcal{F}}(0,T;L^2(G\times S^{d-1}))$. Denote by z (*resp.* Z) the weak limit of $\{z^{n_k}\}_{k=1}^{+\infty}$ (*resp.* $\{Z^{n_k}\}_{k=1}^{+\infty}$).

Let us now show that (z,Z) satisfies (1.6). Let $\gamma\in C([0,T];\mathbb{R})$ such that

$$\frac{d\gamma}{dt}\in L^2(0,T;\mathbb{R}),\ \gamma(0)=0.$$

Set $\gamma_i(t)=\gamma(t)e_i$. Multiplying the equation (2.16) by $\gamma_i(t)$ and employing Itô's formula, we have

$$\begin{aligned}&(z_T^{n_k},\gamma_i(T))_{L^2(G\times S^{d-1})}\\&=\int_0^T\Big(z^{n_k}(t),\frac{d\gamma_i(t)}{dt}\Big)_{L^2(G\times S^{d-1})}dt+\int_0^T\big(z^{n_k}(t),\mathcal{A}^{n_k}\gamma_i(t)\big)_{L^2(G\times S^{d-1})}dt\\&\quad+\int_0^T\big(\mathcal{B}_1^{n_k}(t)z^{n_k}(t)+\mathcal{B}_2^{n_k}(t)z^{n_k}(t),\gamma_i(t)\big)_{L^2(G\times S^{d-1})}dt+\int_0^T\big(\mathcal{B}_3^{n_k}(t)Z^{n_k}(t),\gamma_i(t)\big)_{L^2(G\times S^{d-1})}dt\\&\quad+\int_0^T\big(\mathcal{B}_4^{n_k}(t)z^{n_k}(t)+Z^{n_k}(t),\gamma_i(t)\big)_{L^2(G\times S^{d-1})}dB(t).\end{aligned}\tag{2.18}$$

Letting $k\to+\infty$, we arrive at

$$\begin{aligned}&(z_T,\gamma_i(T))_{L^2(G\times S^{d-1})}\\&=\int_0^T\Big(z(t),\frac{d\gamma_i(t)}{dt}\Big)_{L^2(G\times S^{d-1})}dt+\int_0^T\big(z(t),U\cdot\nabla\gamma_i\big)_{L^2(G\times S^{d-1})}dt\\&\quad+\int_0^T\big(b_1(t)z(t),\gamma_i(t)\big)_{L^2(G\times S^{d-1})}dt+\int_0^T\Big(\big(b_2(t),z(t)\big)_{L^2(S^{d-1})},\gamma_i(t)\Big)_{L^2(G\times S^{d-1})}dt\\&\quad+\int_0^T\big(b_3(t)Z(t),\gamma_i(t)\big)_{L^2(G\times S^{d-1})}dt+\int_0^T\big(b_4(t)z(t)+Z(t),\gamma_i(t)\big)_{L^2(G\times S^{d-1})}dB(t).\end{aligned}\tag{2.19}$$

Thus, for any $\psi\in C_0^\infty(G\times S^{d-1})$, we have

$$\begin{aligned}&\big(z_T,\psi\big)_{L^2(G\times S^{d-1})}\gamma(T)\\&=\int_0^T\big(z(t),\psi\big)_{L^2(G\times S^{d-1})}\frac{d\gamma(t)}{dt}dt+\int_0^T\big(z,U\cdot\nabla\psi\big)_{L^2(G\times S^{d-1})}\gamma dt\\&\quad+\int_0^T\big(b_1(t)z(t),\psi\big)_{L^2(G\times S^{d-1})}\gamma(t)dt+\int_0^T\Big(\big(b_2(t),z(t)\big)_{L^2(S^{d-1})},\psi\Big)_{L^2(G\times S^{d-1})}\gamma(t)dt\\&\quad+\int_0^T\big(b_2(t)Z(t),\psi\big)_{L^2(G\times S^{d-1})}\gamma(t)dt+\int_0^T\big(b_3(t)z(t)+Z(t),\psi\big)_{L^2(G\times S^{d-1})}\gamma(t)dB(t).\end{aligned}\tag{2.20}$$

For any $\tau \in (0, T)$, define γ_ε by

$$\gamma_\varepsilon(s) = \begin{cases} 0, & \text{if } s \le \tau - \dfrac{\varepsilon}{2}, \\ \dfrac{1}{\varepsilon}\left(s - \tau + \dfrac{\varepsilon}{2}\right), & \text{if } \tau - \dfrac{\varepsilon}{2} < s < \tau + \dfrac{\varepsilon}{2}, \\ 1, & \text{if } s \ge \tau + \dfrac{\varepsilon}{2}. \end{cases} \tag{2.21}$$

Substituting γ in the equality (2.20) with γ_ε and letting $\varepsilon \to 0$, we see that for a.e. $(\tau, \omega) \in [0, T] \times \Omega$, it holds that

$$\begin{aligned} &\int_G \int_{S^{d-1}} z_T(x, U)\psi(x, U) dS dx \\ =& \int_G \int_{S^{d-1}} z(\tau, x, U)\psi(x, U) dS dx - \int_\tau^T \int_G \int_{S^{d-1}} z(s, x, U) U \cdot \nabla\psi(x, U) dS dx ds \\ &+ \int_\tau^T \int_G \int_{S^{d-1}} \big[b_1(s, x, U)z(s, x, U) + b_3(s, x, U)Z(s, x, U)\big]\psi(x) dS dx ds \\ &+ \int_\tau^T \int_G \int_{S^{d-1}} \Big[\int_{S^{d-1}} b_2(s, x, U, V)z(s, x, V) dS\Big] dS dx ds \\ &+ \int_\tau^T \int_G \int_{S^{d-1}} \big[b_4(s, x, U)z(s, x, U) + Z(s, x, U)\big]\psi(x, U) dS dx dB(s). \end{aligned} \tag{2.22}$$

This means that (z, Z) satisfies (1.6).

Further, we prove that the inequality (1.7) holds. From the inequality (2.17) and the construction of (z, Z), we get that

$$\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} (|z(t)|^2 + |Z(t)|^2) dS dx dt \le e^{Cr_2} \mathbb{E}\int_G \int_{S^{d-1}} |z_T|^2 dS dx. \tag{2.23}$$

By means of the inequality (2.17) again, we know that for each fixed $t \in [0, T]$, we can find a subsequence $\{n_i\}_{i=1}^\infty$ of $\{n_k\}_{k=1}^\infty$ and a $\tilde{z}(t) \in L^2(\Omega, \mathcal{F}_t, P; L^2(G \times S^{d-1}))$ such that $\{z^{n_i}(t)\}_{i=1}^\infty$ converges to $\tilde{z}(t)$ in $L^2(\Omega, \mathcal{F}_t, P; L^2(G \times S^{d-1}))$ weakly. Then, from (2.17), we get that

$$\int_G \int_{S^{d-1}} |\tilde{z}(t)|^2 dS dx \le e^{Cr_2} \mathbb{E}\int_G \int_{S^{d-1}} |z_T|^2 dS dx, \tag{2.24}$$

where the constant C is independent of $t \in [0, T]$. As a similar argument to obtain the

equality (2.22), we can get

$$
\begin{aligned}
&\int_G\int_{S^{d-1}} z_T(x,U)\psi(x,U)dSdx\\
&=\int_G\int_{S^{d-1}} \tilde z(\tau,x,U)\psi(x,U)dSdx-\int_\tau^T\int_G\int_{S^{d-1}} z(s,x,U)U\cdot\nabla\psi(x,U)dSdxds\\
&\quad+\int_\tau^T\int_G\int_{S^{d-1}}\big[b_1(s,x,U)z(s,x,U)+b_3(s,x,U)Z(s,x,U)\big]\psi(x)dSdxds\\
&\quad+\int_\tau^T\int_G\int_{S^{d-1}}\Big[\int_{S^{d-1}} b_3(s,x,U,V)z(s,x,V)dS\Big]dSdxds\\
&\quad+\int_\tau^T\int_G\int_{S^{d-1}}\big(b_4(s,x,U)z(s,x,U)+Z(s,x,U)\big)\psi(x,U)dSdxdB(s).
\end{aligned}
\tag{2.25}
$$

Therefore, we find that $z(\cdot)=\tilde z(\cdot)$ for a.e. $(t,\omega)\in[0,T]\times\Omega$. From this and the inequality (2.23)–(2.24), we conclude that the inequality (1.7) holds. $\square$

Now we give a proof of Proposition 1.3.

Proof of Proposition 1.3: The proof is almost standard. Here we give it for the sake of completeness. Let

$$X\stackrel{\triangle}{=}\big\{f\in H^1(G\times S^{d-1}):\ f=0 \text{ on } \Gamma_{+S}\big\}.$$

Following the proof of Proposition 1.2 (for this, one needs numerous but small changes), one can show that if $z_T\in L^2(\Omega,\mathcal{F}_T,P;X)$, then the solution

$$(z,Z)\in L^2_{\mathcal F}(0,T;X))\times L^2_{\mathcal F}(0,T;L^2(G\times S^{d-1})).$$

By Itô's formula, we see

$$
\begin{aligned}
&\mathbb{E}|z_T|^2_{L^2(G\times S^{d-1})}-\mathbb{E}|z(0)|^2_{L^2(G\times S^{d-1})}\\
&=-\mathbb{E}\int_0^T\int_G\int_{S^{d-1}} zU\cdot\nabla z dSdxdt+\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\big[2z(b_1z+b_3Z)+(b_4z+Z)^2\big]dSdxdt\\
&\quad+\mathbb{E}\int_0^T\int_G\int_{S^{d-1}} z(t,x,U)\Big[\int_{S^{d-1}} b_2(t,x,U,V)z(t,x,V)dS\Big]dSdxdt.
\end{aligned}
\tag{2.26}
$$

Hence, we find

$$
\begin{aligned}
&-\mathbb{E}\int_0^T\int_{\Gamma_{-S}} U\cdot\nu z^2 d\Gamma_{-S}dt\\
&=\mathbb{E}|z_T|^2_{L^2(G\times S^{d-1})}-\mathbb{E}|z(0)|^2_{L^2(G\times S^{d-1})}+\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\big[2z(b_1z+b_3Z)+(b_4z+Z)^2\big]dSdxdt\\
&\quad+\mathbb{E}\int_0^T\int_G\int_{S^{d-1}} z(t,x,U)\Big[\int_{S^{d-1}} b_2(t,x,U,V)z(t,x,V)dS\Big]dSdxdt\\
&\leq e^{Cr_2}\mathbb{E}|z_T|^2_{L^2(G\times S^{d-1})}.
\end{aligned}
\tag{2.27}
$$

For any $z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G\times S^{d-1}))$, we can find a sequence $\{z_T^{(n)}\}_{n=1}^{\infty} \subset L^2(\Omega, \mathcal{F}_T, P; X)$ such that

$$\lim_{n\to\infty} z_T^{(n)} = z_T \text{ in } L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1})).$$

Hence, we know that the inequality (2.27) also holds for $z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1}))$.

□

Next, we introduce a weighted identity, which will play a key role in the proof of Theorem 1.5. To begin with, we give some functions. Let $0 < c < 1$ such that $cT > 2R$. Put

$$l = \lambda\big(|x|^2 - ct^2\big) \text{ and } \theta = e^l. \tag{2.28}$$

We have the following weighted identity involving θ and l.

Proposition 2.1 *Assume that v is an $H^1(\mathbb{R}^n)\times L^2(S^{d-1})$-valued continuous semi-martingale. Put $p = \theta v$. We have the following equality*

$$\begin{aligned}
&-\theta(l_t + U\cdot\nabla l)p\big[dv + U\cdot\nabla v dt\big]\\
&= -\frac{1}{2}d\big[(l_t + U\cdot\nabla l)p^2\big] - \frac{1}{2}U\cdot\nabla\big[(l_t + U\cdot\nabla l)p^2\big] + \frac{1}{2}\big[l_{tt} + U\cdot\nabla(U\cdot\nabla l)\\
&\quad +2U\cdot\nabla l_t\big]p^2 + \frac{1}{2}(l_t + U\cdot\nabla l)(dp)^2 + (l_t + U\cdot\nabla l)^2p^2.
\end{aligned} \tag{2.29}$$

Proof of Proposition 2.1: By the definition of p, we have

$$\theta(dv + U\cdot\nabla v) = \theta d(\theta^{-1}p) + \theta U\cdot\nabla(\theta^{-1}p) = dp + U\cdot\nabla p - (l_t + U\cdot\nabla l)p.$$

Thus, we know

$$\begin{aligned}
&-\theta(l_t + U\cdot\nabla l)p\big(dv + U\cdot\nabla v\big)\\
&= -(l_t + U\cdot\nabla l)p\big[dp + U\cdot\nabla p - (l_t + U\cdot\nabla l)p\big]\\
&= -(l_t + U\cdot\nabla l)p(dp + U\cdot\nabla p) + (l_t + U\cdot\nabla l)^2p^2.
\end{aligned} \tag{2.30}$$

It is easy to see

$$\left\{\begin{aligned}
&-l_tpdp = -\frac{1}{2}d(l_tp^2) + \frac{1}{2}l_{tt}p^2 + \frac{1}{2}l_t(dp)^2,\\
&-U\cdot\nabla lpdp = -\frac{1}{2}d(U\cdot\nabla lp^2) + \frac{1}{2}(U\cdot\nabla l)_tp^2 + \frac{1}{2}U\cdot\nabla l(dp)^2,\\
&-l_tpU\cdot\nabla p = -\frac{1}{2}U\cdot\nabla(l_tp^2) + \frac{1}{2}U\cdot\nabla l_tp^2,\\
&-U\cdot\nabla lpU\cdot\nabla p = -\frac{1}{2}U\cdot\nabla(U\cdot\nabla lp^2) + \frac{1}{2}U\cdot\nabla(U\cdot\nabla l)p^2.
\end{aligned}\right. \tag{2.31}$$

From (2.30) and (2.31), we obtain the equality (2.29). □

At last, we give a result which will be used to show the negative results, that is, Theorem 1.2–1.4.

Set

$$\eta(t) = \left\{\begin{array}{ll} 1, & \text{if } t \in \big[(1-2^{-2i})T, (1-2^{-2i-1})T\big), \quad i = 0, 1, \cdots,\\ -1, & \text{otherwise}\end{array}\right.$$

Lemma 2.1 *[11, Lemma 2.1] Let* $\xi = \int_0^T \eta(t)dB(t)$. *It is impossible to find* $(\varrho_1, \varrho_2) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R})$ *and* $x \in \mathbb{R}$ *with*

$$\lim_{t \to T} \mathbb{E}|\varrho_2(t) - \varrho(T)|^2 = 0,$$

such that

$$\xi = x + \int_0^T \varrho_1(t)dt + \int_0^T \varrho_2(t)dB(t). \tag{2.32}$$

3 Proof of Theorem 1.5

This section is devoted to proving Theorem 1.5. We complete the mission by employing a global Carleman estimate for the equation (1.5).

Proof of Theorem 1.5: To start with, applying Proposition 2.1 to the equation (1.5) with $v = z$, integrating (2.29) on $(0,T) \times G \times S^{d-1}$, and taking mathematical expectation, we get that

$$\begin{aligned}
&-2\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)z(dz + U \cdot \nabla z dt)dSdxdt \\
&= \lambda\mathbb{E}\int_G \int_{S^{d-1}} (cT - 2U \cdot x)\theta^2(T)z^2(T)dSdx + \lambda\mathbb{E}\int_G \int_{S^{d-1}} (cT + 2U \cdot x)\theta^2(0)z^2(0)dSdx \\
&+2c\lambda\mathbb{E}\int_0^T \int_{\Gamma_{-S}} U \cdot \nu\big[c(T-2t) - 2U \cdot x\big]\theta^2 z^2 d\Gamma_{-S}dt + 2(1-c)\lambda\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2 z^2 dSdxdt \\
&+\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)(b_4 z + Z)^2 dSdxdt + 2\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)^2 z^2 dSdxdt.
\end{aligned} \tag{3.1}$$

By virtue of that z solves the equation (1.5), we see

$$\begin{aligned}
&-2\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)z(dz + U \cdot \nabla z dt)dSdxdt \\
&= 2\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)z\Big(b_1 z + \int_{S^{d-1}} b_2(t,x,U,V)z(t,x,V)dS + b_3 Z\Big)dSdxdt \\
&\leq \mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)^2 z^2 dxdt + 3\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2(b_1^2 z^2 + b_3^2 Z^2)dSdxdt \\
&\quad +3\mathbb{E}\int_0^T \int_G \int_{S^{d-1}} \theta^2\Big|\int_{S^{d-1}} b_2(t,x,U,V)z(t,x,V)dS\Big|^2 dSdxdt.
\end{aligned} \tag{3.2}$$

This, together with the equality (3.1), implies that

$$
\begin{aligned}
&\lambda\mathbb{E}\int_G\int_{S^{d-1}}(cT-2U\cdot x)\theta^2(T)z^2(T)dSdx+\lambda\mathbb{E}\int_G\int_{S^{d-1}}(cT+2U\cdot x)\theta^2(0)z^2(0)dSdx\\
&+2(1-c)\lambda\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2z^2dSdxdt+\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(l_t+U\cdot\nabla l)(b_4z+Z)^2dSdxdt\\
&+\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(l_t+U\cdot\nabla l)^2z^2dSdxdt\\
\le\ & 3\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(b_1^2z^2+b_3^2Z^2)dSdxdt-2c\lambda\mathbb{E}\int_0^T\int_{\Gamma_{-S}}U\cdot\nu\big[c(T-2t)-2U\cdot x\big]\theta^2z^2d\Gamma_{-S}dt\\
&+3\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2\Big|\int_{S^{d-1}}b_2(t,x,U,V)z(t,x,V)dS\Big|^2dSdxdt.
\end{aligned}
\tag{3.3}
$$

Since

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(l_t+U\cdot\nabla l)(b_4z+Z)^2dSdxdt\\
&\le\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(l_t+U\cdot\nabla l)^2z^2dSdxdt+\frac{1}{2}\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(b_4^4+2b_4^2)z^2dSdxdt\\
&\quad+\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(l_t+U\cdot\nabla l+2)Z^2dSdxdt,
\end{aligned}
$$

by means of the inequality (3.3), we find

$$
\begin{aligned}
&\lambda\mathbb{E}\int_G\int_{S^{d-1}}(cT-2U\cdot x)\theta^2(T)z^2(T)dSdx+\lambda\mathbb{E}\int_G\int_{S^{d-1}}(cT+2U\cdot x)\theta^2(0)z^2(0)dSdx\\
&+2(1-c)\lambda\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2z^2dSdxdt-3\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(b_1^2+b_4^4+b_4^2)z^2dSdxdt\\
&-3|b_2|^2_{L^\infty_{\mathcal{F}}(\Omega;L^\infty(0,T;C(\overline{G}\times S^{d-1}\times S^{d-1})))}\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2z^2dSdxdt\\
&\le 3\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2\big[b_3^2+2+\lambda x-c\lambda t\big]Z^2dSdxdt\\
&\quad-2c\lambda\mathbb{E}\int_0^T\int_{\Gamma_{-S}}U\cdot\nu\big[c(T-2t)-2U\cdot x\big]\theta^2z^2d\Gamma_{-S}dt.
\end{aligned}
\tag{3.4}
$$

Owing to $|x|<2R$, we know that

$$
\left\{
\begin{aligned}
&(cT-2R)\mathbb{E}\int_G\int_{S^{d-1}}\theta^2(T)z^2(T)dSdx\le\mathbb{E}\int_G\int_{S^{d-1}}\theta^2(T)(cT-U\cdot x)z^2(T)dSdx,\\
&(cT-2R)\mathbb{E}\int_G\int_{S^{d-1}}\theta^2(0)z^2(0)dSdx\le\mathbb{E}\int_G\int_{S^{d-1}}\theta^2(0)(cT+U\cdot x)z^2(0)dSdx.
\end{aligned}
\right.
\tag{3.5}
$$

Taking

$$\lambda_1 = \frac{3}{2(1-c)}\big(|b_1|^2_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1}))} + |b_2|^2_{L^\infty_{\mathcal{F}}(\Omega;L^\infty(0,T;C(\overline{G}\times S^{d-1}\times S^{d-1})))}$$
$$+|b_4|^4_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1}))} + |b_4|^2_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G\times S^{d-1}))}\big),$$

for any $\lambda \geq \lambda_1$, it holds that

$$\begin{aligned}
&3\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2(b_1^2+b_4^4+b_4^2)z^2dSdxdt \\
&+3|b_2|^2_{L^\infty_{\mathcal{F}}(\Omega;L^\infty(0,T;C(\overline{G}\times S^{d-1}\times S^{d-1})))}\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2z^2dSdxdt \\
\leq\ & 2(1-c)\lambda\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2z^2dSdxdt.
\end{aligned} \tag{3.6}$$

From (3.4)–(3.6), and noting that $cT > 2R$, we find that

$$\begin{aligned}
&\mathbb{E}\int_G\int_{S^{d-1}}\theta^2(T,x)z^2(T,x)dSdx \\
\leq\ & \mathbb{E}\int_0^T\int_G\int_{S^{d-1}}\theta^2\big(b_3^2+2\lambda x-2c\lambda t\big)Z^2dSdxdt \\
&-2c\lambda\mathbb{E}\int_0^T\int_{\Gamma_{-S}}U\cdot\nu\big[c(T-2t)-2U\cdot x\big]\theta^2z^2d\Gamma_{-S}dt.
\end{aligned} \tag{3.7}$$

By the definition of θ, we have

$$e^{-c\lambda T^2} \leq \theta \leq e^{4\lambda R^2}.$$

This, together with the inequality (3.7), shows that

$$\begin{aligned}
&e^{-2c\lambda T^2}\mathbb{E}\int_G\int_{S^{d-1}}z_T^2dSdx \\
\leq\ & Ce^{8\lambda R^2}\left\{\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}Z^2dSdxdt + \mathbb{E}\int_0^T\int_{\Gamma_{-S}}U\cdot\nu z^2d\Gamma_{-S}dt\right\},
\end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned}
&\mathbb{E}\int_G\int_{S^{d-1}}z_T^2dx \\
\leq\ & Ce^{2c\lambda_1T^2+8\lambda_1R^2}\Big\{\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}Z^2dSdxdt + \mathbb{E}\int_0^T\int_{\Gamma_{-S}}U\cdot\nu z^2d\Gamma_{-S}dt\Big\} \\
\leq\ & e^{Cr_2^2}\Big\{\mathbb{E}\int_0^T\int_G\int_{S^{d-1}}Z^2dxdt + \mathbb{E}\int_0^T\int_{\Gamma_{-S}}U\cdot\nu z^2d\Gamma_{-S}dt\Big\}.
\end{aligned} \tag{3.9}$$

This completes the proof. $\square$

4 Proof of Theorem 1.1

This section is addressed to a proof of Theorem 1.1.

Proof of Theorem 1.1: Since the system (1.1) is linear, we only need to show that the attainable set A_T at time T with initial datum $y(0) = 0$ is $L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1}))$, that is, for any $y_1 \in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1}))$, we can find a pair of control

$$(u, v) \in L^2_{\mathcal{F}}(0, T; L^2_w(L^2(\Gamma_{-S}))) \times L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$$

such that the solution to the system (1.1) satisfies $y(T) = y_1$ in $L^2(G \times S^{d-1})$, P-a.s. We achieve this goal by duality argument.

Let $b_1 = -a_1$, $b_2 = -a_2$ and $b_3 = -a_3$ and $b_4 = 0$ in the equation (1.5). We introduce the following linear subspace of $L^2_{\mathcal{F}}(0, T; L^2_w(L^2(\Gamma_{-S}))) \times L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$:

$$\mathcal{X} \triangleq \Big\{ \big(-z|_{\Gamma_{-S}}, Z \big) \;\Big|\; (z, Z) \text{ solves the equation (1.5) with some } z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1})) \Big\}$$

and define a linear functional $\mathcal{L}$ on $\mathcal{X}$ as follows:

$$\mathcal{L}(z(\cdot, 0), Z) = \mathbb{E} \int_G \int_{S^{d-1}} y_1 z_T dS dx - \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} z f dS dx dt.$$

From Theorem 1.5, we see that $\mathcal{L}$ is a bounded linear functional on $\mathcal{X}$. By means of the Hahn-Banach theorem, $\mathcal{L}$ can be extended to be a bounded linear functional on the space $L^2_{\mathcal{F}}(0, T; L^2_w(L^2(\Gamma_{-S}))) \times L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$. For simplicity, we still use $\mathcal{L}$ to denote this extension. Now, by Riesz Representation theorem, we know there is a pair of random fields

$$(u, v) \in L^2_{\mathcal{F}}(0, T; L^2_w(L^2(\Gamma_{-S}))) \times L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$$

so that

$$\begin{aligned} & \mathbb{E} \int_G \int_{S^{d-1}} y_1 z_T dS dx - \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} z f dS dx dt \\ & = -\mathbb{E} \int_0^T \int_{\Gamma_{-S}} U \cdot \nu z u d\Gamma_{-S} dt + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} v Z dS dx dt. \end{aligned} \tag{4.1}$$

We claim that this pair of random fields (u, v) is exactly the control we need. Indeed, by means of Itô's formula, we have

$$\begin{aligned} & \mathbb{E} \int_G \int_{S^{d-1}} y(T, \cdot) z_T dS dx \\ = & \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} (-zU \cdot \nabla y + a_1 y z + f z) dS dx dt + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} (a_3 y Z + v Z) dS dx dt \\ & + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \Big(\int_{S^{d-1}} a_2 y dS \Big) z dS dx dt + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} (-U \cdot \nabla z y - a_1 y z - a_3 y Z) dS dx dt \\ & - \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \Big(\int_{S^{d-1}} a_2 y dS \Big) z dS dx dt. \end{aligned} \tag{4.2}$$

Hence, it holds that

$$
\begin{aligned}
&\mathbb{E}\int_G\int_{S^{d-1}} y(T,\cdot)z_T dSdx - \mathbb{E}\int_0^T\int_G\int_{S^{d-1}} zfdSdxdt \\
&= -\mathbb{E}\int_0^T\int_{\Gamma_{-S}} U\cdot\nu zud\Gamma_{-S}dt + \mathbb{E}\int_0^T\int_G\int_{S^{d-1}} vZdSdxdt.
\end{aligned} \tag{4.3}
$$

From (4.1) and (4.3), we see

$$
\mathbb{E}\int_G\int_{S^{d-1}} y_1 z_T dSdx = \mathbb{E}\int_G\int_{S^{d-1}} y(T,\cdot)z_T dSdx. \tag{4.4}
$$

Since z_T can be arbitrary element in $L^2(\Omega,\mathcal{F}_T,P;L^2(G\times S^{d-1}))$, from the equality (4.4), we get $y(T)=y_1$ in $L^2(G\times S^{d-1})$, P-a.s. □

5 Proof of the lack of exact controllability

The purpose of this section is to give a proof of Theorem 1.2–1.4. In order to present the key idea in the simplest way, we only consider a very special case of the system (1.1), that is, $G=(0,1)$, $a_1=0$, $a_2=0$, $a_3=1$, $a_4=0$ and $f=0$. The argument for the general case is very similar.

Proof of Theorem 1.2: We first show that if $v=0$ in $L^2_{\mathcal{F}}(0,T;L^2(0,1))$, then the system (1.1) is not exactly controllable at any time T. In this case, the system we study is as follows:

$$
\begin{cases}
dy + y_x dt = ydB(t) & \text{in } (0,T)\times(0,1),\\
y(t,0) = u(t) & \text{on } (0,T)\times\{0\},\\
y(0) = y_0 & \text{in } (0,1).
\end{cases} \tag{5.1}
$$

It is easy to see the solution of the system (5.1) is

$$
y(t,x) = \begin{cases}
e^{B(t)-\frac{t}{2}} y_0(x-t), & \text{if } x>t,\\
e^{B(t)-\frac{t}{2}} \hat{u}(t-x), & \text{if } x\le t.
\end{cases} \tag{5.2}
$$

Here $\hat{u}(t)=e^{-B(t)+\frac{t}{2}}u(t)$. We first deal with the case $T\ge 1$. Choose a

$$
\tilde{y}\in L^2(\Omega,\mathcal{F}_T,P;L^2(0,1)) \text{ such that } \chi_{(\frac{1}{2},1)}\tilde{y}\notin L^2(\Omega,\mathcal{F}_{T-\frac{1}{2}},P;L^2(0,1)).
$$

Set $y_1=\chi_{(\frac{1}{2},1)}e^{B(T)-\frac{T}{2}}\tilde{y}$. Then we know that for any control $u\in L^2_{\mathcal{F}}(0,T;\mathbb{R})$, it holds that $y_1\neq y(T)$ in $L^2(\Omega,\mathcal{F}_T,P;L^2(0,1))$. Otherwise we get $\chi_{(T-1,T-\frac{1}{2})}u=\chi_{(\frac{1}{2},1)}\tilde{y}$ in $(\frac{1}{2},1)$, which means that $\chi_{(T-1,T-\frac{1}{2})}u\notin L^2(\Omega,\mathcal{F}_{T-\frac{1}{2}},P;L^2(0,1))$. This leads to a contradiction.

Now we treat the case $T<L$. The idea is the same as the proof for $T\ge L$. Choose a

$$
\bar{y}\in L^2(\Omega,\mathcal{F}_T,P;L^2(0,1)) \text{ such that } \chi_{(1-T,1)}\bar{y}\notin L^2(\Omega,\mathcal{F}_0,P;L^2(0,1)).
$$

Set $y_1 = \chi_{(1-T,1)}e^{B(T)-\frac{T}{2}}\tilde{y}$. Then we have $y_1 \neq y(T)$ in $L^2(\Omega, \mathcal{F}_T, P; L^2(0,1))$, for any control $u \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$. If not, we find $\chi_{(1-T,1)}y_0 = \chi_{(1-T,1)}\tilde{y} \notin L^2(\Omega, \mathcal{F}_0, P; L^2(0,1))$, which leads to a contradiction.

Now we are in a position to prove Theorem 1.2 for $u \equiv 0$. In this case, the system (1.1) reads as

$$
\begin{cases}
dy + y_x dt = (1+f)dB(t) & \text{in } (0,T)\times(0,1),\\
y(t,0) = 0 & \text{on } (0,T)\times\{0\},\\
y(0) = y_0 & \text{in } (0,1).
\end{cases}
\tag{5.3}
$$

Since the system (5.3) is linear, we only need to show that the attainable set A_T at time T for initial datum $y_0 = 0$ is not $L^2(\Omega, \mathcal{F}_T, P; L^2(0,1))$. The solution of the system (1.4) is

$$
y(T) = S(T)y_0 + \int_0^T S(T-s)\big[1+f(s)\big]dB(s). \tag{5.4}
$$

Here $\{S(t)\}_{t\geq 0}$ is the semigroup introduced in Section 2. One can see [2, Chapter 6] for establishing (5.4).

From (5.4), we find $\mathbb{E}(y(T)) = \mathbb{E}(S(T)y_0)$. Thus, if we choose a $y_1 \in L^2(\Omega, \mathcal{F}_T, P; L^2(0,1))$ such that $\mathbb{E}(y_1) \neq 0$, then y_1 is not in A_T, which completes the proof. $\square$

Proof of Theorem 1.3: Put

$$
\mathcal{V} \overset{\triangle}{=} \{v:\ v \in L^2_{\mathcal{F}}(0,T;L^2(0,1)),\ v = 0 \text{ in } (0,T)\times G_0\}.
$$

Let ξ be given by Lemma 2.1. Choose a $\psi \in C_0^\infty(G_0)$ such that $|\psi|_{L^2(G)} = 1$ and set $y_T = \xi\psi$. We will show that y_T cannot be attained for any $y_0 \in \mathbb{R}$, $u \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and $v \in \mathcal{V}$. This goal is achieved by contradiction argument. If there exist a $u \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and a $v \in \mathcal{V}$ such that the corresponding solution $y(\cdot)$ satisfies $y(T) = y_T$, then by Itô's formula, we obtain

$$
\begin{aligned}
\xi &= \int_G y_T\psi dx\\
&= \int_G y_0\psi dx - \int_0^T\int_G \psi y_x dxdt + \int_0^T\int_G \psi(y+v)dxdB(t)\\
&= \int_G y_0\psi dx + \int_0^T\Big(\int_G \psi_x y dx\Big)dt + \int_0^T\Big(\int_G \psi y dx\Big)dB(t).
\end{aligned}
\tag{5.5}
$$

It is clear that both $\displaystyle\int_G \psi_x y dx$ and $\displaystyle\int_G \psi y dx$ belong to $L^2_{\mathcal{F}}(0,T;\mathbb{R})$. Further,

$$
\lim_{t\to T}\mathbb{E}\Big|\int_G \psi y(t)dx - \int_G \psi y(T)dx\Big|^2 = \lim_{t\to T}\mathbb{E}\Big|\int_G \psi\big[y(t)-y(T)\big]dx\Big|^2 = 0.
$$

These, together with (5.5), contradict Lemma 2.1. $\square$

Proof of Theorem 1.4: The proof is similar to the one for Theorem 1.3.

Let ξ be given by Lemma 2.1. Choose a $\psi \in C_0^\infty(G)$ such that $|\psi|_{L^2(G)} = 1$ and set $y_T = \xi\psi$. We will show that y_T cannot be attained for any $y_0 \in \mathbb{R}$, $u \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$

and $\ell \in L^2_{\mathcal{F}}(0,T;L^2(0,1))$. It is done by contradiction argument too. If there exist a $u \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and an $\ell \in L^2_{\mathcal{F}}(0,T;L^2(0,1))$ such that the corresponding solution $y(\cdot)$ satisfies $y(T) = y_T$, then by Itô's formula, we obtain

$$\begin{aligned}
\xi &= \int_G y_T \psi dx \\
&= \int_G y_0 \psi dx - \int_0^T \int_G \psi(y_x - \ell) dx dt + \int_0^T \int_G \psi y dx dB(t) \\
&= \int_G y_0 \psi dx + \int_0^T \Big(\int_G \psi_x y dx + \int_G \psi \ell dx \Big) dt + \int_0^T \Big(\int_G \psi y dx \Big) dB(t).
\end{aligned} \tag{5.6}$$

It is clear that both $\int_G \psi_x y dx + \int_G \psi \ell dx$ and $\int_G \psi y dx$ belong to $L^2_{\mathcal{F}}(0,T;\mathbb{R})$. Further,

$$\lim_{t\to T} \mathbb{E}\Big| \int_G \psi y(t) dx - \int_G \psi y(T) dx \Big|^2 = \lim_{t\to T} \mathbb{E}\Big| \int_G \psi \big[y(t) - y(T) \big] dx \Big|^2 = 0.$$

These, together with (5.6), contradict Lemma 2.1. $\square$

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